

# Small noise approximation of center manifolds for stochastic dynamical systems \*

Jian Ren<sup>1</sup>, Zhongkai Guo<sup>2</sup>, Xianming Liu<sup>3</sup> and Xiangjun Wang<sup>4</sup>  
School of Mathematics and Statistics

Huazhong University of Science and Technology, Wuhan, 430074, China

Email: 1. renjian0371@gmail.com, 2. nj4102008@gmail.com,

3. mathliuxm@yahoo.cn, 4. xjwang@mail.hust.edu.cn

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Institute for Pure and Applied Mathematics, University of California, Los Angeles, CA 90095, USA

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## Abstract

This paper provides a small noise approximation for local random center manifolds of a class of stochastic dynamical systems in Euclidean space. An example is presented to illustrate the method.

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## 1 Introduction

Center manifolds, together with stable and unstable manifolds, provide geometric structures that help understand nonlinear and stochastic dynamics. For stochastic dynamical systems, it is difficult to visualize or depict center manifolds. In the paper, we derive a small noise approximation for random center manifolds.

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We consider a system of nonlinear random differential equations (RDE) on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,

$$\frac{du}{dt} = A(\theta_t \omega)u + F(\theta_t \omega, u), \quad (1)$$

where  $u \in \mathbb{R}^n$ , and  $\theta_t : \Omega \rightarrow \Omega$  is an ergodic flow that preserves the probability measure  $\mathbb{P}$ . Assume that 0 is a fixed point, i.e.,  $F(\theta_t \omega, 0) = 0$  for all  $t$  and all  $\omega$ .

This RDE system is usually coming from a stochastic differential equation (SDE) system, by a stationary coordinate transform [5, 7].

Boxler [2, 3] proved a random center manifold theorem, and also considered reduction to the center manifold. Power series, in terms of variables on the center manifold, approximations are also available in [3, 8]. Sun [7] provided a small noise approximation for random stable or unstable manifolds.

In this present paper, we derive a small noise approximation of random center manifolds, for the stochastic system (1) where the linear operator  $A(\theta_t \omega)$  has exponential trichotomy property.

## 2 Exponential trichotomy

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space with an ergodic flow  $\theta_t$ . A measurable map  $\phi(t, \omega, u) : \mathbb{R} \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called a random dynamical system (RDS) if it satisfies the following conditions

- (i)  $\phi(0, \omega, u) = u$ ;
- (ii)  $\phi(t + s, \omega, u) = \phi(t, \theta_s \omega, u)\phi(s, \omega, u)$ , for all  $t, s \in \mathbb{R}$  and all  $\omega \in \Omega$ .

The latter is the so-called cocycle property.

For a linear RDE system in  $\mathbb{R}^n$  of the form

$$\frac{du}{dt} = A(\theta_t \omega)u, \quad (2)$$

where matrix  $A(\theta_t \omega)$  satisfies the integrability condition that  $A \in \mathbf{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ . The integrability of  $A(\theta_t \omega)$  implies that the RDE system (2) generates a linear cocycle  $\Phi(t, \omega)$ , which is the fundamental matrix  $e^{\int_0^t A(\theta_\tau \omega) d\tau}$ . Then the multiplicative ergodic theorem (MET), in [1, Chapter 3], yields that we have Lyapunov exponents  $\lambda_1 > \lambda_2 > \dots > \lambda_r$ , with  $r \leq n$ , together with  $r$  Osledets subspaces  $E_i(\omega)$ , so that  $\mathbb{R}^n$  decomposes into the direct sum:

$$\mathbb{R}^n = E_1(\omega) \oplus E_2(\omega) \oplus \dots \oplus E_r(\omega). \quad (3)$$

All the subspaces  $E_i(\omega)$  are measurable and are random invariant subspaces of  $\Phi(t, \omega)$ , i.e. for  $i = 1, 2, \dots, r$ ,

$$\Phi(t, \omega)E_i(\omega) = E_i(\theta_t \omega), \quad \text{for all } t \in \mathbb{R}, \mathbb{P} - a.s. \quad (4)$$

We define stable, center and unstable subspaces

$$E^s(\omega) \triangleq \oplus_{\lambda_i < 0} E_i(\omega), \quad (5)$$

$$E^c(\omega) \triangleq \oplus_{\lambda_i = 0} E_i(\omega), \quad (6)$$

$$E^u(\omega) \triangleq \oplus_{\lambda_i > 0} E_i(\omega). \quad (7)$$

Therefore,

$$\mathbb{R}^n = E^s(\omega) \oplus E^c(\omega) \oplus E^u(\omega),$$

and  $\Phi(t, \omega)$  can be decomposed as

$$\Phi(t, \omega) = \Phi_s(t, \omega) \oplus \Phi_c(t, \omega) \oplus \Phi_u(t, \omega).$$

Denote

$$\lambda_s \triangleq \max_{\lambda_i < 0} \lambda_i < 0, \quad \lambda_u \triangleq \min_{\lambda_i > 0} \lambda_i > 0,$$

then

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |\Phi(t, \omega)v| = \lambda,$$

with

$$\begin{aligned} \lambda &= \lambda_s, \quad \text{for } v \in E^s(\omega) \setminus \{0\}, \\ \lambda &= \lambda_u, \quad \text{for } v \in E^u(\omega) \setminus \{0\}, \\ \lambda &= 0, \quad \text{for } v \in E^c(\omega) \setminus \{0\}. \end{aligned}$$

Taking a positive number  $\gamma < \frac{1}{2} \min\{|\lambda_s|, \lambda_u\}$ , the MET leads to the following estimation [2, Lemma 4.2]:

$$\begin{aligned} |\Phi_s(t, \omega)v| &\leq e^{(\lambda_s + \gamma)t} v, \quad v \in E^s, \quad t \geq 0, \\ |\Phi_c(t, \omega)v| &\leq e^{\gamma|t|} v, \quad v \in E^c, \quad t \in \mathbb{R}, \\ |\Phi_u(t, \omega)v| &\leq e^{(\lambda_u - \gamma)t} v, \quad v \in E^u, \quad t < 0. \end{aligned} \quad (8)$$

Similar to the definition of exponential dichotomy in [5], the cocycle  $\Phi(t, \omega)$  with the conditions (3), (4) and (8) hold is said to satisfy the exponential trichotomy.

For the nonlinear RDE system

$$\frac{du}{dt} = A(\theta_t \omega)u + F(\theta_t \omega, u), \quad (9)$$

we assume that the nonlinear term  $F$  is Lipschitz continuous on  $\mathbb{R}^n$ , that is,

$$|F(u_1) - F(u_2)| \leq L_F |u_1 - u_2|,$$

with the Lipschitz constant  $L_F > 0$ . In the next section, we consider the random center manifold for this system.

If the nonlinear term  $F$  is only locally Lipschitz, then we let  $F^{(R)}(u) = \chi_R(u)F(u)$ , where  $\chi_R(u)$  is a cut-off function. Thus  $F^{(R)}$  is globally Lipschitz with Lipschitz constant  $RL_F$ ; see [4, Lemma 4.1]. In this case we would obtain a local random center manifold.

By using the fundamental function

$$\Phi(t, \omega) = e^{\int_0^t A(\theta_\tau \omega) d\tau},$$

the solution of the random differential system (9) can be interpreted as

$$u(t, \omega, u_0) = \Phi(t, \omega)u_0 + \int_0^t \Phi(t - \tau, \omega)F(\theta_\tau \omega, u(\tau)) d\tau.$$

This equation has a unique measurable solution due to the Lipschitz continuity of  $F$ , and the solution mapping  $(t, \omega, u_0) \mapsto u(t, \omega, u_0)$  generates a random dynamical system  $\phi(t, \omega, u_0)$ . Moreover, the Jacobian mapping  $D_{u_0}u(t, \omega, u_0) = \Phi(t, \omega)$  satisfies the exponential trichotomy.

### 3 Random center manifold

In this section, we consider random center manifolds for a SDE system with small multiplicative noise. We will recall a random center manifold theorem [2] and then approximate the random center manifold under small noise, in the case of non-positive Lyapunov exponents. This is done in deterministic case in [4] and [6, Chapter 3].

We introduce the definition of random center manifolds.

**Definition 1** (Random Center Manifold). *A random set  $M(\omega)$  is called a random center manifold for a random dynamical system  $\phi(t, \omega, x)$ , if it satisfies the following conditions*

(i) *It is an invariant set, i.e.  $\phi(t, \omega, M(\omega)) \subset M(\theta_t \omega)$  for all  $t$ .*

(ii) *It can be represented as a graph of a mapping from the center subspace to its complement, i.e. there is a mapping  $h^c(\omega, \cdot) : E^c \rightarrow E^u \oplus E^s$ , such that  $M(\omega) = \{(v, h^c(\omega, v)) : v \in E^c\}$ , where  $h^c(\omega, 0) = 0$ ,  $h^c(\cdot, v)$  is measurable for every  $v \in E^c$  and the tangency condition  $Dh^c(\omega, 0) = 0$  holds. The center manifold  $M(\omega)$  is often denoted as  $M^c(\omega)$ .*

*It is called a Lipschitz center manifold if the mapping  $h^c(\omega, \cdot) : E^c \rightarrow E^u \oplus E^s$  is Lipschitz and the tangency condition is absent. We call  $M^c(\omega)$  a local center manifold if it is a graph of the mapping  $\chi_R(v) h^c(\omega, \cdot)$ , where  $\chi_R(v)$  is a cut-off function.*

For the nonlinear SDE system defined in the previous section, Boxler [2, 3] has shown that there exists a (local) random center manifold, as the graph of a mapping  $h^c$  that satisfies the following Liapunov-Perron equation

$$h^c(\omega, x^c) = \left( \int_{-\infty}^0 \Phi_s^{-1}(\tau, \omega) F^s \left( \theta_\tau \omega, (\phi^c(\tau, \omega, x^c, h^c(\omega, x^c))), h^c(\theta_\tau \omega, \phi^c(\tau, \omega, x^c, h^c(\omega, x^c)))) \right) d\tau, \right. \\ \left. \int_0^\infty \Phi_u^{-1}(\tau, \omega) F^u \left( \theta_\tau \omega, (\phi^c(\tau, \omega, x^c, h^c(\omega, x^c))), h^c(\theta_\tau \omega, \phi^c(\tau, \omega, x^c, h^c(\omega, x^c)))) \right) d\tau \right), \quad (10)$$

for  $x^c \in E^c$  and  $\omega \in \Omega$  where  $F^s, F^u$  are respectively the projection of  $F$  to the stable and unstable subspaces,  $\phi^c$  is the projection of  $\phi$  to the center subspace.

### 3.1 Center manifold of a system with small noise

We consider a SDE system in Stratonovich form

$$\begin{cases} \dot{x} = A^c x + f^c(x, y) + (\varepsilon x^\top \circ \dot{W}_t^1)^\top, & x \in \mathbb{R}^n, \\ \dot{y} = A^s y + f^s(x, y) + (\varepsilon y^\top \circ \dot{W}_t^2)^\top, & y \in \mathbb{R}^m, \end{cases} \quad (11)$$

where  $^\top$  indicates the transpose of a vector or a matrix. In this system,  $A^c$  and  $A^s$  are respectively  $n \times n$  and  $m \times m$  matrixes. The nonlinear functions  $f^c(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $f^s(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  are  $C^1$ -smooth and Lipschitz with Lipschitz constant  $L_f$ , i.e.  $f^c(x, y)$  and  $f^s(x, y)$  satisfy

$$|f^c(x_1, y_1) - f^c(x_2, y_2)| \leq L_f (|x_1 - x_2| + |y_1 - y_2|),$$

and

$$|f^s(x_1, y_1) - f^s(x_2, y_2)| \leq L_f (|x_1 - x_2| + |y_1 - y_2|),$$

where  $|x|$  is the Euclidean norm of  $x \in \mathbb{R}^n$  or  $x \in \mathbb{R}^m$ . Usually,  $f^c, f^s$  are locally Lipschitz continuous and in that case we would get a local random center manifold. The noise intensity  $\varepsilon$  is a small positive parameter. Moreover,  $\{W_t^i : t \in \mathbb{R}, i = 1, 2\}$  are  $n \times n$  and  $m \times m$  matrixes, respectively, with two-sided scalar Wiener process  $W_t$  as principal diagonal elements and all other elements being zero.

For matrixes  $A^c$  and  $A^s$ , we make the following assumption.

**(H):** There are positive constants  $\beta, \gamma$  and  $K$ , satisfying  $\beta > \gamma \geq 0$  and  $K > 0$ , such that for every  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$ , the following exponential estimates hold

$$|e^{A^c t} x|_{\mathbb{R}^n} \leq K e^{\gamma|t|} |x|_{\mathbb{R}^n}, \quad t \in \mathbb{R}; \quad |e^{A^s t} y|_{\mathbb{R}^m} \leq K e^{-\beta|t|} |y|_{\mathbb{R}^m}, \quad t \geq 0.$$

To facilitate random dynamical systems approach, we convert (11) into a system of random differential equations (RDE). To this end, consider linear stochastic differential equations

$$dz_i + z_i dt = dW_t^i, \quad i = 1, 2. \quad (12)$$

Each of these equations has a unique stationary solution

$$z_i(\omega) = \int_{-\infty}^0 e^\tau dW_\tau^i, \quad i = 1, 2.$$

Moreover,

$$z_i(\theta_t \omega) = \int_{-\infty}^t e^{\tau-t} dW_\tau^i, \quad i = 1, 2.$$

Thus, the linear differential equations,

$$dZ_i + Z_i dt = \varepsilon dW_t^i, \quad i = 1, 2. \quad (13)$$

have unique stationary solutions  $Z_i(\omega) = \varepsilon z_i(\omega)$  and  $Z_i(\theta_t \omega) = \varepsilon z_i(\theta_t \omega)$  for  $i = 1, 2$ .

Note that  $z_i(\omega)$ ,  $z_i(\theta_t \omega)$ ,  $Z_i(\omega)$  and  $Z_i(\theta_t \omega)$  for  $i = 1, 2$  are all principals diagonals matrixes, with the principal diagonals values  $\int_{-\infty}^0 e^\tau dW_\tau$ ,  $\int_{-\infty}^t e^{\tau-t} dW_\tau$ ,  $\varepsilon \int_{-\infty}^0 e^\tau dW_\tau$  and  $\varepsilon \int_{-\infty}^t e^{\tau-t} dW_\tau$  respectively.

We introduce a random transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} := \mathcal{V}_\varepsilon(\omega, x, y) = \begin{pmatrix} e^{-\varepsilon z_1(\omega)} x \\ e^{-\varepsilon z_2(\omega)} y \end{pmatrix}.$$

Under this transformation, the SDE system (11) is converted into a RDE system

$$\begin{cases} \dot{X} = A^c X + \varepsilon z_1 X + F^c(X, Y), & X \in \mathbb{R}^n, \\ \dot{Y} = A^s Y + \varepsilon z_2 Y + F^s(X, Y), & Y \in \mathbb{R}^m. \end{cases} \quad (14)$$

with

$$F^c(X, Y) = e^{-\varepsilon z_1(\theta_t \omega)} f^c(e^{\varepsilon z_1(\theta_t \omega)} X, e^{\varepsilon z_2(\theta_t \omega)} Y),$$

and

$$F^s(X, Y) = e^{-\varepsilon z_2(\theta_t \omega)} f^s(e^{\varepsilon z_1(\theta_t \omega)} X, e^{\varepsilon z_2(\theta_t \omega)} Y),$$

being Lipschitz functions with the Lipschitz constant  $L_f$ .

For every  $\eta > 0$ , we define a product Banach Space  $C_\eta := C_\eta^1 \times C_\eta^2$ , where

$$C_\eta^1 = \left\{ \varphi : \mathbb{R} \rightarrow \mathbb{R}^n : \varphi \text{ is continuous and } \sup_{t \in \mathbb{R}} |e^{-\eta|t|} e^{-\varepsilon \int_0^t z_1(\theta_r \omega) dr} \varphi(t)|_{\mathbb{R}^n} < \infty \right\},$$

$$C_\eta^2 = \left\{ \varphi : \mathbb{R} \rightarrow \mathbb{R}^m : \varphi \text{ is continuous and } \sup_{t \in \mathbb{R}} |e^{-\eta|t|} e^{-\varepsilon \int_0^t z_1(\theta_r \omega) dr} \varphi(t)|_{\mathbb{R}^m} < \infty \right\},$$

with norm

$$|\varphi(t)|_{C_\eta^1} = \sup_{t \in \mathbb{R}} |e^{-\eta|t|} e^{-\varepsilon \int_0^t z_1(\theta_r \omega) dr} \varphi(t)|_{\mathbb{R}^n},$$

and

$$|\phi(t)|_{C_\eta^2} = \sup_{t \in \mathbb{R}} |e^{-\eta|t|} e^{-\varepsilon \int_0^t z_1(\theta_r \omega) dr} \varphi(t)|_{\mathbb{R}^m}.$$

respectively. Furthermore, the norm in  $C_\eta$  is

$$|(X, Y)|_{C_\eta} = |X|_{C_\eta^1} + |Y|_{C_\eta^2}.$$

The integral form of the differential equation (14) can be written as

$$\begin{cases} X(t) = e^{A^c t + \varepsilon \int_0^t z_1(\theta_r \omega) dr} X(0) + \int_0^t e^{A^c(t-\tau) + \varepsilon \int_\tau^t z_1(\theta_r \omega) dr} F^c(X, Y) d\tau, \\ Y(t) = \int_{-\infty}^t e^{A^s(t-\tau) + \varepsilon \int_\tau^t z_2(\theta_r \omega) dr} F^s(X, Y) d\tau. \end{cases} \quad (15)$$

Due to the Lipschitz condition of  $F^c$  and  $F^s$  in (14) and the measurability of  $z_i(\omega)$ ,  $i = 1, 2$ , the RDE system (14) has a unique solution. Therefore, the solution mapping generates a RDS. By the assumption **(H)** on  $A^c$ ,  $A^s$  and the properties of  $z_i(\omega)$ ,  $i = 1, 2$ , as in [5], the exponential trichotomy holds (with  $E^u$  empty). We thus have following result [2, 3].

**Theorem 2.** Assume that **H** holds and that  $\gamma < \eta < \beta$  satisfy the gap condition  $\frac{KL_f}{\eta - \gamma} + \frac{KL_f}{\beta - \eta} < 1$ . Then there exists a random center manifold  $\tilde{\mathcal{H}}^\varepsilon(\omega) = (\tilde{\xi}, \tilde{H}^\varepsilon(\omega, \tilde{\xi}))$  for (14), where  $\tilde{H}^\varepsilon(\omega, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a Lipschitz function.

### 3.2 Small noise approximation of random center manifolds

We now approximate the above random center manifold under small noise intensity  $\varepsilon$ . Denote the corresponding deterministic center manifold (when  $\varepsilon = 0$ ) and the random center manifold as

$$\tilde{\mathcal{H}}^d(\tilde{\xi}) = \{(\tilde{\xi}, \tilde{H}^d(\tilde{\xi}))\}, \quad (16)$$

and

$$\tilde{\mathcal{H}}^\varepsilon(\omega, \tilde{\xi}) = \{(\tilde{\xi}, \tilde{H}^\varepsilon(\omega, \tilde{\xi})) = (\tilde{\xi}, \tilde{H}^d(\tilde{\xi}) + \varepsilon \tilde{H}^1(\omega, \tilde{\xi}) + \mathcal{O}(\varepsilon^2))\}, \quad (17)$$

where  $\tilde{H}^d(\cdot)$  and  $\tilde{H}^1(\omega, \cdot)$  are locally defined on  $\mathbb{R}^n$ . Denoting  $x(0) = \xi \in \mathbb{R}^n$ , as in [5], the center manifold  $\mathcal{H}^\varepsilon(\omega, \xi)$  can be converted back to the center manifold for the original stochastic system (11),

$$\begin{aligned} \mathcal{H}^\varepsilon(\omega, \xi) &= \{(\xi, H^\varepsilon(\omega, \xi)) : \xi \in \mathbb{R}^n\} = \mathcal{V}_\varepsilon^{-1}(\cdot, \tilde{\mathcal{M}}^\varepsilon(\omega)) \\ &= \{(e^{\varepsilon z_1(\omega)} \tilde{\xi}, e^{\varepsilon z_2(\omega)} \tilde{H}^\varepsilon(\omega, \tilde{\xi}))\} = \{(\xi, e^{\varepsilon z_2(\omega)} \tilde{H}^\varepsilon(\omega, e^{-\varepsilon z_1(\omega)} \xi))\} \\ &= \{(\xi, \tilde{H}^d(\xi) + \varepsilon \tilde{H}^1(\omega, \xi) + \varepsilon z_2(\omega) \tilde{H}^d(\xi) - \varepsilon \tilde{H}_\xi^d(\xi) z_1(\omega) \xi + \mathcal{O}(\varepsilon^2))\}. \end{aligned} \quad (18)$$

To approximate the center manifold, we expand

$$X(t) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \cdots, \quad \text{with } X(0) = \tilde{\xi} = X_0(0), \quad \tilde{\xi} \in \mathbb{R}^n,$$

and

$$Y(t) = Y_0(t) + \varepsilon Y_1(t) + \varepsilon^2 Y_2(t) + \cdots, \quad (19)$$

with

$$Y(0) = \tilde{H}^\varepsilon(\omega, \tilde{\xi}) = \tilde{H}^d(\tilde{\xi}) + \varepsilon \tilde{H}^1(\omega, \tilde{\xi}) + \mathcal{O}(\varepsilon^2).$$

Noting that

$$e^{-\varepsilon z_i} = 1 - \varepsilon z_i + \frac{\varepsilon^2 z_i^2}{2!} + \cdots, \quad \text{for } i = 1, 2,$$

we have,

$$\begin{aligned} F^c(X(t), Y(t)) &= e^{-\varepsilon z_1(\theta_t \omega)} f^c(e^{\varepsilon z_1(\theta_t \omega)} X(t), e^{\varepsilon z_2(\theta_t \omega)} Y(t)) \\ &= f^c(X_0(t), Y_0(t)) + \varepsilon \left\{ f_x^c(X_0(t), Y_0(t)) [X_1(t) + z_1(\theta_t \omega) X_0(t)] \right. \\ &\quad \left. + f_y^c(X_0(t), Y_0(t)) [Y_1(t) + z_2(\theta_t \omega) Y_0(t)] - z_1(\theta_t \omega) f^c(X_0(t), Y_0(t)) \right\} + \mathcal{O}(\varepsilon^2), \end{aligned}$$

and

$$\begin{aligned} F^s(X(t), Y(t)) &= e^{-\varepsilon z_2(\theta_t \omega)} f^s(e^{\varepsilon z_1(\theta_t \omega)} X(t), e^{\varepsilon z_2(\theta_t \omega)} Y(t)) \\ &= f^s(X_0(t), Y_0(t)) + \varepsilon \left\{ f_x^s(X_0(t), Y_0(t)) [X_1(t) + z_1(\theta_t \omega) X_0(t)] \right. \\ &\quad \left. + f_y^s(X_0(t), Y_0(t)) [Y_1(t) + z_2(\theta_t \omega) Y_0(t)] - z_2(\theta_t \omega) f^s(X_0(t), Y_0(t)) \right\} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Inserting these expansions into (14) and matching the terms with the same powers of  $\varepsilon$ , we get at the 0-th order

$$\begin{pmatrix} \dot{X}_0(t) \\ \dot{Y}_0(t) \end{pmatrix} = \begin{pmatrix} A^c & 0 \\ 0 & A^s \end{pmatrix} \begin{pmatrix} X_0(t) \\ Y_0(t) \end{pmatrix} + \begin{pmatrix} f^c(X_0(t), Y_0(t)) \\ f^s(X_0(t), Y_0(t)) \end{pmatrix},$$

which can be expressed as

$$\begin{pmatrix} X_0(t) \\ Y_0(t) \end{pmatrix} = \exp \left\{ \begin{pmatrix} A^c & 0 \\ 0 & A^s \end{pmatrix} t \right\} \begin{pmatrix} \tilde{\xi} \\ \tilde{H}_d(\tilde{\xi}) \end{pmatrix} + \int_0^t \exp \left\{ \begin{pmatrix} A^c & 0 \\ 0 & A^s \end{pmatrix} (t - \tau) \right\} \begin{pmatrix} f^c(X_0(s), Y_0(s)) \\ f^s(X_0(s), Y_0(s)) \end{pmatrix} d\tau. \quad (20)$$



Furthermore, at the first order in  $\varepsilon$ ,

$$\begin{pmatrix} \dot{X}_1(t) \\ \dot{Y}_1(t) \end{pmatrix} = \left[ \begin{pmatrix} A^c & 0 \\ 0 & A^s \end{pmatrix} + \begin{pmatrix} f_x^c(X_0(t), Y_0(t)) & f_y^c(X_0(t), Y_0(t)) \\ f_x^s(X_0(t), Y_0(t)) & f_y^s(X_0(t), Y_0(t)) \end{pmatrix} \right] \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \\ + \begin{pmatrix} f_x^c z_1(\theta_t \omega) X_0(t) + z_1(\theta_t \omega) X_0(t) + f_y^c z_2(\theta_t \omega) Y_0(t) - z_1(\theta_t \omega) f^c \\ f_x^s z_1(\theta_t \omega) X_0(t) + z_2(\theta_t \omega) Y_0(t) + f_y^s z_2(\theta_t \omega) Y_0(t) - z_2(\theta_t \omega) f^s \end{pmatrix}, \quad (21)$$

which can be rewritten as

$$\begin{pmatrix} X_1(t) \\ Y_1(t) \end{pmatrix} = \exp \left\{ \begin{pmatrix} A^c & 0 \\ 0 & A^s \end{pmatrix} t + \int_0^t \begin{pmatrix} f_x^c(X_0(s), Y_0(s)) & f_y^c(X_0(s), Y_0(s)) \\ f_x^s(X_0(s), Y_0(s)) & f_y^s(X_0(s), Y_0(s)) \end{pmatrix} ds \right\} \begin{pmatrix} X_1(0) \\ Y_1(0) \end{pmatrix} \\ + \int_0^t \exp \left\{ \begin{pmatrix} A^c & 0 \\ 0 & A^s \end{pmatrix} (t - \tau) - \int_s^t \begin{pmatrix} f_x^c & f_y^c \\ f_x^s & f_y^s \end{pmatrix} dr \right\} \begin{pmatrix} f_x^c z_1 X_0 + z_1 X_0 + f_y^c z_2 Y_0 - z_1 f^c \\ f_x^s z_1 X_0 + z_2 Y_0 + f_y^s z_2 Y_0 - z_2 f^s \end{pmatrix} d\tau. \quad (22)$$

Hence,

$$\begin{aligned} \tilde{H}^\varepsilon(\omega, \tilde{\xi}) &= \int_{-\infty}^0 e^{-A^s \tau + \varepsilon \int_\tau^0 z_2(\theta_r \omega) dr} F^s(X(\tau), Y(\tau)) d\tau \\ &= \int_{-\infty}^0 e^{-A^s \tau} \left( 1 + \varepsilon \int_\tau^0 z_2(\theta_r \omega) dr + \frac{\varepsilon^2 (\int_\tau^0 z_2(\theta_r \omega) dr)^2}{2!} + \dots \right) \left\{ f^s(X_0(\tau), Y_0(\tau)) \right. \\ &\quad + \varepsilon \left[ f_x^s(X_0(\tau), Y_0(\tau)) (X_1(\tau) + z_1(\theta_\tau \omega) X_0(\tau)) + f_y^s(X_0(\tau), Y_0(\tau)) (Y_1(\tau) + z_2(\theta_s \omega) Y_0(\tau)) \right. \\ &\quad \left. \left. - z_2(\theta_\tau \omega) f^s(X_0(\tau), Y_0(\tau)) \right] \right\} d\tau + \mathcal{O}(\varepsilon^2) \\ &= \int_{-\infty}^0 e^{-A^s \tau} f^s(X_0(\tau), Y_0(\tau)) d\tau + \varepsilon \left\{ \int_{-\infty}^0 e^{-A^s \tau} \left[ \int_\tau^0 z_2(\theta_r \omega) dr f^s(X_0(\tau), Y_0(\tau)) \right. \right. \\ &\quad + f_x^s(X_0(\tau), Y_0(\tau)) (X_1(\tau) + z_1(\theta_\tau \omega) X_0(\tau)) + f_y^s(X_0(\tau), Y_0(\tau)) (Y_1(\tau) + z_2(\theta_\tau \omega) Y_0(\tau)) \\ &\quad \left. \left. - z_2(\theta_\tau \omega) f^s(X_0(\tau), Y_0(\tau)) \right] d\tau \right\} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Therefore, matching with (19), we get

$$\tilde{H}^d(\tilde{\xi}) = \int_{-\infty}^0 e^{-A^s \tau} f^s(X_0(\tau), Y_0(\tau)) d\tau, \quad (23)$$

and

$$\begin{aligned}\tilde{H}^1(\omega, \tilde{\xi}) &= \int_{-\infty}^0 e^{-A^s \tau} \left\{ \int_{\tau}^0 z_2(\theta_r \omega) dr f^s(X_0(\tau), Y_0(\tau)) + f_x^s(X_0(\tau), Y_0(\tau)) [X_1(\tau) + z_1(\theta_r \omega) X_0(\tau)] \right. \\ &\quad \left. + f_y^s(X_0(\tau), Y_0(\tau)) [Y_1(\tau) + z_2(\theta_r \omega) Y_0(\tau)] - z_2(\theta_r \omega) f^s(X_0(\tau), Y_0(\tau)) \right\} d\tau.\end{aligned}\quad (24)$$

Moreover, by (18), the center manifold for the original stochastic system (11) is

$$\mathcal{H}^\varepsilon(\omega) = \{(\xi, H^\varepsilon(\omega, \xi))\} = \{(\xi, H^d(\xi) + \varepsilon H^1(\omega, \xi) + \mathcal{O}(\varepsilon^2))\},$$

with

$$H^d(\xi) = \tilde{H}^d(\xi), \quad (25)$$

and

$$H^1(\omega, \xi) = \tilde{H}^1(\omega, \xi) + z_2(\omega) \tilde{H}^d(\xi) - \tilde{H}_\xi^d(\xi) z_1(\omega) \xi. \quad (26)$$

We summarize the above approximation result in the following theorem.

**Theorem 3** (Approximation of a random center manifold). *Assume that (H) holds and that  $\eta$  satisfies  $\gamma < \eta < \beta$  and  $\frac{KL_f}{\eta - \gamma} + \frac{KL_g}{\beta - \eta} < 1$ . Then there exists a center manifold  $\mathcal{H}^\varepsilon(\omega) = \{(\xi, H^\varepsilon(\omega, \xi))\}$  for the stochastic system (11), where  $H^\varepsilon(\omega, \xi) = H^d(\xi) + \varepsilon H^1(\omega, \xi) + \mathcal{O}(\varepsilon^2)$ , with  $H^d(\xi)$  and  $H^1(\omega, \xi)$  in (25) and (26), respectively.*

We now look at an example of the approximation of random center manifolds.

*Example 1.* Consider a SDE system

$$\begin{cases} \dot{x} = a^c x + \varepsilon x \circ \dot{W}_t, & x \in \mathbb{R}, \\ \dot{y} = -y - x^2 + \varepsilon y \circ \dot{W}_t, & y \in \mathbb{R}. \end{cases} \quad (27)$$

Here  $A^c = a^c$ ,  $A^s = -1$ ,  $f^c(x, y) = 0$ ,  $f^s(x, y) = -x^2$ ,  $z(\omega) = \int_{-\infty}^0 e^\tau dW_\tau$  and  $z(\theta_t \omega) = \int_{-\infty}^t e^{\tau-t} dW_\tau = e^{-t} z(\omega) + e^{-t} \int_0^t e^\tau dW_\tau$ . We restrict this system on a bounded disk containing the origin. The new system satisfies the assumption (H). In this way, we obtain a local center manifold and we now consider its small noise approximation.

We transform this SDE system into a RDE system,

$$\begin{cases} \dot{X}(t) = a^c X + \varepsilon z(\theta_t \omega) X, & x \in \mathbb{R}, \\ \dot{Y}(t) = -Y + \varepsilon z(\theta_t \omega) Y - e^{-\varepsilon z(\theta_t \omega)} (e^{\varepsilon z(\theta_t \omega)} X)^2, & y \in \mathbb{R}. \end{cases} \quad (28)$$

Denote  $a = a^c + \varepsilon z(\theta_t \omega)$ ,  $b = -1 + \varepsilon z(\theta_t \omega)$ , and  $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ . By the property

$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t z(\theta_\tau \omega) d\tau = 0$  in [5], we have the Lyapunov exponents

$$\begin{cases} \lambda_c = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |e^{\int_0^t A(\tau) d\tau} u| = a^c = 0, & \text{when } u = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \\ \lambda_s = -1 < 0, & \text{when } u = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}. \end{cases} \quad (29)$$

By (20) and (23), we get

$$X_0(t) = \tilde{\xi}, \quad \tilde{H}^d(\tilde{\xi}) = \int_{-\infty}^0 e^\tau (-X_0^2) d\tau = -\tilde{\xi}^2,$$

and

$$Y_0(t) = e^{-t} \tilde{H}^d(\tilde{\xi}) - \tilde{\xi}^2(1 - e^{-t}) = -\tilde{\xi}^2.$$

From (22) and (24), we obtain

$$X_1(t) = \tilde{H}^1(\omega, \tilde{\xi}) + \tilde{\xi} \int_0^t z(\theta_\tau \omega) d\tau,$$

and

$$\begin{aligned} \tilde{H}^1(\omega, \tilde{\xi}) &= \int_{-\infty}^0 e^\tau \left[ \int_\tau^0 z(\theta_r \omega) dr (-X_0^2(\tau)) - 2X_0(\tau)(X_1(\tau) + z(\theta_\tau \omega)X_0(\tau)) + z(\theta_\tau \omega)X_0^2(\tau) \right] d\tau \\ &= -2\tilde{\xi} \tilde{H}^1(\omega, \tilde{\xi}) - \tilde{\xi}^2 \int_{-\infty}^0 e^\tau \left[ \int_\tau^0 z(\theta_r \omega) dr + 2 \left( \int_0^\tau z(\theta_r \omega) dr + z(\theta_\tau \omega) \right) - z(\theta_\tau \omega) \right] d\tau, \end{aligned}$$

thus

$$\tilde{H}^1(\omega, \tilde{\xi}) = \frac{\tilde{\xi}^2}{1 + 2\tilde{\xi}} \left[ \int_{-\infty}^0 e^\tau \int_\tau^0 z(\theta_r \omega) dr d\tau - \int_{-\infty}^0 e^\tau z(\theta_\tau \omega) d\tau \right] = 0. \quad (30)$$

By (25) and (26)

$$H^d(\xi) = \tilde{H}^d(\xi) = -\xi^2,$$

and

$$\begin{aligned}
H^1(\omega, \xi) &= \tilde{H}^1(\omega, \xi) + z(\omega)\tilde{H}^d(\xi) - \tilde{H}_\xi^d(\xi)z(\omega)\xi \\
&= 0 - z(\omega)\xi^2 + 2z(\omega)\xi^2 \\
&= \xi^2 \int_{-\infty}^0 e^\tau dW_\tau.
\end{aligned}$$

Therefore, we have an approximation for center manifold of (27) for  $\varepsilon$  sufficiently small

$$\mathcal{H}^\varepsilon(\omega) = \{(\xi, H^\varepsilon(\omega, \xi))\},$$

with

$$H^\varepsilon(\omega, \xi) = -\xi^2 + \varepsilon \xi^2 \int_{-\infty}^0 e^\tau dW_\tau + \mathcal{O}(\varepsilon^2).$$

The center manifold of the corresponding deterministic system

$$\begin{cases} \dot{x} = 0, \\ \dot{y} = -y - x^2. \end{cases} \tag{31}$$

is the graph of the function  $h^c(\xi) = -\xi^2$ . A few realizations of the approximate center manifold, together with the deterministic center manifold, are shown in Figure 1.

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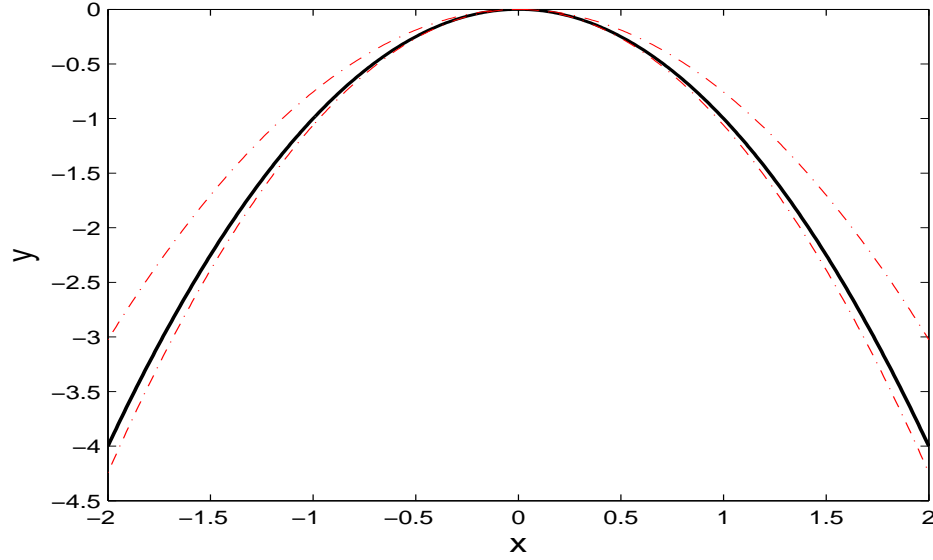


Figure 1: Approximate center manifold of system (27):  $\varepsilon = 0$  (no noise, black or solid curve) v.s.  $\varepsilon = 0.2$  (Red or dash-dot curve, with two realizations).

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